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Coalition Formation in Games with Externalities

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Coalition Formation in Games with Externalities

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Abstract

This paper studies an extensive form game of coalition formation with random proposers in a situation where coalitions impose externalities on other players. It is shown that an agreement will be reached without delay provided that any set of coalitions profit from merging. Even under this strong condition, the formation of the grand coalition is not guaranteed. Therefore, the resulting coalition structure will not necessarily be efficient.

The results of this model are compared with the related work of Ray and Vohra (GEB, 1999), which assumes that players move in a predetermined order. The game with random proposers tends to give a large advantage to the proposer, whereas the game with a rule of order tends to favour the responders and may not capture the competition between players. The game with random proposers yields more efficient results for some specific classes of games. However, the results of the two games cannot be ranked in general in terms of efficiency.

Keywords: coalition formation, externalities, partition function, random proposers.

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1 Introduction

In most economic situations, the payo¤ of a coalition depends on what other coalitions form, that is, there are externalities between coalitions. A function assigning to each coalition a payo¤ depending on the whole coalition structure is called a partition function (see Thrall and Lucas, (1963)). However, standard cooperative game theory is not based on the partition function but on the characteristic function, which assigns a payo¤ to each coalition regardless of the actions of outsiders. In order to derive a characteristic function from a situation with externalities, players are assumed to have conjectures about how the rest of the players will organize themselves given that a coalition forms. These conjectures are usually pessimistic (players fear the worst) or optimistic (players expect the best)[1]. Thus, when a coalition forms, the rest of the players are expected to partition themselves so as to either minimize or maximize the payo¤ of the coalition, regardless of their own interest.

An alternative to indiscriminated optimism or pessimism is to use an extensive form game of coalition formation (together with the concept of subgame perfection) in order to allow a coalition to predict the reaction of the outsiders as an equilibrium reaction, so that conjectures are consistent (see Bloch (1996) and Ray and Vohra (1999)). Of course, extensive form games have the drawback that the reaction of the outsiders may heavily depend on the details of the extensive form game.

Because the details of the extensive form game matter, it is useful to compare di¤erent extensive form games. This paper coincides with Bloch (1996) and Ray and Vohra (1999) in using an extensive form game to predict coalition formation, but it di¤ers in the concrete extensive form.

Ray and Vohra (1999) extend the model of coalitional bargaining of Chatterjee et al (1993) to games with externalities. Both models are natural generalizations of the Rubinstein (1982) two-player alternating o¤ers bargaining game. A distinctive feature of these models is that players respond to proposals according to a predetermined rule of order and the …rst player to reject a proposal automatically becomes the next proposer. Competition between the players is then very limited, as two players may continuously make o¤ers and countero¤ers to each other without any other player having the opportunity to step in.

The assumption that players, by making o¤ers to each other, may exclude other players from the negotiations may not always be appropriate. Muthoo

[1] For a recent illustration of this approach, see Funaki and Yamato (1999).

(1999) refers to this question in his book:

> In modelling this three player (one seller-two buyers) bargaining situation, one should allow for the possibility that the two buyers may consider forming a coalition, and then bargain with the seller over the sale (...). At the same time, perhaps while the buyers are negotiating the terms and structure of their coalition, the seller may have the opportunity to approach one of the two buyers and negotiate to sell the house to her - a strategy intended to prevent the formation of the buyers' coalition. Clearly, such factors need to be considered when modelling bargaining situations with three or more players.

The present paper attempts to address this issue by considering an alternative model of coalitional bargaining. This model has been studied by Binmore (1987) for two players, Baron and Ferejohn (1989) for symmetric majority games, and Okada (1996) for characteristic function games. This paper considers the extension to partition function bargaining. The distinctive feature of this model is that a player who rejects an o¤er does not automatically become the next proposer. Instead, proposers are selected randomly by Nature. By giving less power to the responders, this model incorporates competition between the players: any player may have an opportunity to "step in" with a proposal during the negotiations.

As it is common in coalitional bargaining games, binding contracts can be written over the division of the coalitional payo¤, so that both the coalition structure and the payo¤ division are determined endogenously. A player that has proposed a coalition and a division of payo¤s is committed to his proposal, and no other forward commitments are possible.

Since the payo¤ of a coalition depends on the whole coalition structure, players may have to take their coalition formation decisions without knowing what the payo¤ of the coalition will be. This is not a problem since in equilibrium the probabilities of each coalition structure can be calculated from the strategies of the players and players anticipate the expected coalitional payo¤ by the usual backward induction argument.

In principle, the proposal could assign a di¤erent payo¤ division to each possible coalition structure. Players could then calculate their expected payo¤s given the equilibrium probabilities of each coalition structure, and decide whether to accept or reject the proposal on the basis of these expected payo¤s. Since it is only expected payo¤s that matter, we adopt the simplifying convention that the proposer o¤ers a …xed payo¤ to the rest of players in the coalition (he "buys the right to represent them") and thus becomes the residual claimant.

The partition function is assumed to be fully cohesive, that is, a merger of two or more coalitions is always (weakly) profitable for a fixed partition of the remaining players. Two possible extensive form games are considered, differing on the source of friction in the bargaining process: a model with discounting of payoffs, and a model with an exogenous probability of breakdown after a proposal is rejected[2]. We will consider the limiting case in which frictions are arbitrarily small. The solution concept is stationary perfect equilibrium.

The main focus of the analysis is on the efficiency (in the sense of aggregate payoff maximization) of the outcome. Two relevant properties are immediate agreement and formation of the grand coalition. These properties are sufficient for efficiency given the assumptions, and also necessary if players discount the future and the grand coalition is the only coalition structure that maximizes aggregate payoffs.

Agreement is always immediate in the model with discounting. Since time is not valuable in the model with breakdown probability, it is not surprising that delay can arise in equilibrium. However, delay in this context does not imply a loss of efficiency; furthermore, we show that a delayed agreement must be efficient. We also show that delay is not a generic phenomenon and that a player who creates delay in equilibrium by making an unacceptable proposal is playing a "weak" strategy in the sense that he could get the same payoff by making an acceptable proposal to the grand coalition. One can derive the following "rule of thumb" from this result: if the grand coalition cannot arise in equilibrium, neither can agreement be delayed.

Two natural sufficient conditions are derived for the grand coalition to arise in equilibrium given that all players have the same probability of becoming proposers. For the model with discounting, the grand coalition must have the highest per capita payoff over all coalition structures. For the model with probability of breakdown, the grand coalition must have the highest per capita gain (with respect to the situation with no coalitions) over all coalition structures.

Given an equilibrium of the extensive form game, one can construct a characteristic function game by assigning to each coalition its expected equilibrium payoff. We show that, for any equilibrium such that the grand coalition forms with probability one, expected equilibrium payoffs must lie in the core of this characteristic function game.

Finally, the game with random proposers is compared with the game with a

[2] For a comparison of these two models in the context of two-player bargaining, see Binmore et al (1986).

rule of order studied in Ray and Vohra (1999). The game with random proposers guarantees immediate agreement for fully cohesive games, unlike the game with a rule of order. In general, the two games cannot be ranked in terms of efficiency. However, in very specific cases, like symmetric games without externalities in which only one coalition with positive value can form, the outcome of the game with random proposers is at least as efficient as the outcome of the game with a rule of order. In this particular case, the higher efficiency of the random proposers game is achieved due to the advantage of the proposer (which does not disappear in the limit when the frictions of the bargaining process vanish), thus there is a trade-off between efficiency and distribution.

The rest of the paper is organized as follows. Section 2 presents the two versions of the random proposers model. Sections 3 and 4 are devoted to the efficiency properties of the equilibrium. Section 5 relates efficient equilibria to the core of the characteristic function game derived from the equilibrium strategies. Section 6 compares the game with random proposers studied in this paper to the game with a rule of order in Ray and Vohra (1999). Section 7 concludes.

2 The Model

2.1 The partition function

Let $N = \{1, 2, \ldots, n\}$ be the set of players. The non-empty subsets of N are called coalitions. A coalition structure $\pi := \{S_1, \ldots, S_m\}$ is a partition of N into coalitions, hence it satisfies

$$S_j \cap S_k = \emptyset \text{ if } j \neq k; \quad \cup_{j=1}^{m} S_j = N. \tag{1}$$

The set of all coalition structures is denoted by $\Pi(N)$. For any subset T of N, the set of partitions of T is denoted by $\Pi(T)$ with typical element π_T.

A pair (S, π) with $S \in \pi$ is called an embedded coalition. The set of all embedded coalitions is denoted by $E(N)$.

A partition function ϕ assigns a real number to each embedded coalition (S, π), thus $\phi : E(N) \longrightarrow \mathbb{R}$. The value $\phi(S, \pi)$ represents the payoff of coalition S given that coalition structure π forms.

Given a coalition structure $\pi = \{S_1, \ldots, S_m\}$ and a partition function ϕ, we will denote the m-dimensional vector $(\phi(S_i, \pi))_{i=1}^{m}$ by $\vec{\phi}(S_1, \ldots, S_m)$. It will be sometimes convenient to write down the partition function in terms of $\vec{\phi}$. We will also denote $\phi(N, \{N\})$ by $\phi(N)$ and the partition of N into singletons by $\langle N \rangle$.

Analogously, the partition of any set $S \subset N$ into singletons will be denoted by $\langle S \rangle$. We will economize on brackets and suppress the commas between elements of the same coalition. Thus, we will write $\varphi(\{i\};\{\{i\};\{j;k\}\})$ as $\varphi(i;\{i;jk\})$ and $\mu(\{i\};\{j;k\})$ as $\mu(i;jk)$:

Definition 1 A cooperative game in partition function form (a partition function game) is a pair $(N;\varphi)$, where N denotes the set of players and $\varphi : E(N) \longrightarrow \mathbb{R}$:

Definition 2 A partition function game $(N;\varphi)$ is positive if

$$\varphi(S;\pi) \geq 0 \text{ for all } (S;\pi) \text{ and } \varphi(S;\pi) > 0 \text{ for all } (S;\pi),\ |S| \geq 2:$$

Definition 3 A partition function game $(N;\varphi)$ is superadditive if for all $\pi \in \Pi(N)$, S_i, $S_j \in \pi$, $S_i \neq S_j$ it holds that

$$\varphi(S_i \cup S_j;(\pi \setminus \{S_i;S_j\}) \cup \{S_i \cup S_j\}) \geq \varphi(S_i;\pi) + \varphi(S_j;\pi):$$

Superadditivity means that a merger of any two coalitions is weakly profitable for a given partition of the remaining players.

Definition 4 A partition function game $(N;\varphi)$ is cohesive if

$$\varphi(N) \geq \sum_{S \in \pi} \varphi(S;\pi) \text{ for all } (S;\pi) \in E(N):$$

Cohesiveness means that total payoffs are maximized when players form the grand coalition. Thus, starting from an arbitrary partition, a merger of all coalitions to form the grand coalition is always weakly profitable. If any merger of coalitions is weakly profitable for any given partition of the remaining players, the partition function is fully cohesive.

Definition 5 A partition function game $(N;\varphi)$ is fully cohesive if for all $(\pi;S) \in E(N)$ and for all $\pi_S \in \Pi(S)$ it holds that

$$\varphi(S;\pi) \geq \sum_{T \in \pi_S} \varphi(T;(\pi \setminus \{S\}) \cup \pi_S):$$

Notice that cohesiveness and superadditivity are independent properties, both of them weaker than full cohesiveness. The following examples illustrate these three properties.

Example 1 A game that is superadditive but not cohesive

$N = \{1, 2, 3\}$

$\pi(1, 2, 3) = (3, 3, 3)$

$\pi(ij, k) = (7, 0)$

$\pi(123) = 8$

Even though any merger of two coalitions is profitable, total payoffs are maximized when all players remain singletons. The externality that a two-player merger imposes on the outsider firm is stronger than the internal gain, thus the game is not cohesive.

Example 2 A game that is cohesive but not superadditive

$N = \{1, 2, 3\}$

$\pi(1, 2, 3) = (1, 1, 1)$

$\pi(ij, k) = (0, 3)$

$\pi(N) = 5$

Example 3 A game that is cohesive and superadditive but not fully cohesive

$N = \{1, 2, 3, 4\}$

$\pi(1, 2, 3, 4) = (3, 3, 3, 3)$

$\pi(ij, k, l) = (7, 0, 0)$

$\pi(ijk, l) = (8, 0)$

$\pi(ij, kl) = (2, 2)$

$\pi(N) = 15$

The merger of any two coalitions is profitable and the grand coalition achieves the highest payoff, but the merger of three singletons is unprofitable.

Example 4 A game that is fully cohesive

$N = \{1, 2, 3, 4\}$

$\pi(1, 2, 3, 4) = (1, 1, 1, 1)$

$\pi(ij, k, l) = (3, 0, 0)$

$\pi(ijk, l) = (3, 0)$

$\pi(ij, kl) = (1, 1)$

$\pi(N) = 4$

This example shows that full cohesiveness does not imply that going from a finer to a coarser partition will always improve payoffs. Starting from the situation where all players are singletons, full cohesiveness implies that any two of them would profit from merging. Moreover, starting from a partition of the type $\{ij, k, l\}$, full cohesiveness implies that k and l would profit from merging. Nevertheless, starting from the partition $\{i, j, k, l\}$ players need not benefit by forming the coarser partition $\{ij, kl\}$, as the example shows.

Example 4 also shows that a partition function may be fully cohesive while at the same time the formation of coalitions can only reduce aggregate payoffs.

We will assume in this paper that the partition function is fully cohesive; for some of the results we will also assume it to be positive.

2.2 The noncooperative bargaining game

The noncooperative bargaining procedure described in this section was introduced by Binmore (1987) in the context of two-player bargaining games, and later extended by Baron and Ferejohn (1989) to symmetric majority games, and by Okada (1996) to characteristic function games. This paper considers the extension to partition function games.

We will consider two versions of the bargaining game, depending on the source of friction: the game with discounting and the game with breakdown probability.

2.2.1 The game with discounting

Given the underlying partition function game (N, φ), bargaining proceeds as follows: Nature selects a player randomly to be the proposer according to the probability vector $\mu := (\mu_i)_{i \in N}$, where $\mu_i > 0\ \forall i \in N$ and $\sum_{i \in N} \mu_i = 1$. This probability vector is called a protocol. If $\mu_i = \frac{1}{n}$ for all i it is called symmetric or egalitarian protocol. The proposer i makes a proposal $(S, x^{S\setminus\{i\}})$ where S is a coalition to which i belongs and $x^{S\setminus\{i\}} = (x_j^{S\setminus\{i\}})_{j \in S\setminus\{i\}}$ is a payoff vector. The jth component $x_j^{S\setminus\{i\}}$ represents the payoff player j will receive provided that all players in S accept the proposal and that a coalition structure is formed. Thus, we can think of i as "buying the right to represent coalition S"[3]. Given a proposal, the rest of players in S (called responders) accept or reject sequentially (the order

[3] Given that players are not financially constrained and that the solution concept is stationary perfect equilibrium (see below), it does not matter whether the proposal specifies a fixed payoff for each responder or a division of the coalitional payoff contingent on the final coalition structure. The reason is that, given a proposal to form S with a contingent payoff division, players can compute expected payoffs using the probability that each coalition structure will form (stationarity implies that these probabilities are independent of the payoff division in S), and they will accept or reject on the basis of expected payoffs.

does not affect the results). If all players in S accept, S is formed and bargaining continues between players in $N \setminus S$ provided $|N \setminus S| > 1$. A player j will then be selected to be the proposer with probability 0 if he belongs to S and with probability $\frac{\theta_j}{\sum_{k \in N \setminus S} \theta_k}$ if he does not[4]. If at least one player in S rejects, the game proceeds to the next period in which Nature selects a new proposer according to θ. Players are risk neutral and share a discount factor $\delta < 1$. We will think of δ as being close to 1.

When a coalition structure π is formed[5], each coalition $S \in \pi$ gets $\varphi(S; \pi)$. Coalitions of more than one player share this payoff in the following way: the player i whose proposal to form S was accepted pays $x_j^{S \setminus \{i\}}$ to responder j and gets the residual payoff $\varphi(S; \pi) - \sum_{j \in S \setminus \{i\}} x_j^{S \setminus \{i\}}$. This payoff may be negative (we assume players are not financially constrained). Thus, if a coalition structure is formed at time t, each player in S other than i receives $\delta_j^{t-1} x_j^{S \setminus \{i\}}$ and i receives $\delta^{t-1}[\varphi(S; \pi) - \sum_{j \in S \setminus \{i\}} x_j^{S \setminus \{i\}}]$. If no coalition structure is formed, all players receive zero.

We assume a period elapses after a proposal is rejected, but not after a coalition is formed. One may alternatively assume that a period elapses in both cases without affecting the results in any essential way.

A (pure) strategy for player i is a sequence $\sigma_i = (\sigma_i^t)_{t=1}^{\infty}$, where σ_i^t, the tth round strategy of player i, prescribes

(i) A proposal $(S; x^{S \setminus \{i\}})$.

(ii) A response function assigning "yes" or "no" to all possible proposals of the other players.

Notice that since no time elapses after a coalition is formed there may be several "stages" at time t, each of them with a smaller set of remaining players than the previous.

The solution concept is stationary perfect equilibrium. A stationary perfect equilibrium is a subgame perfect equilibrium with the property that the strategies of the players depend only on the set of coalitions that have already formed, $\pi_{N \setminus T}$, and the current proposal.

An equilibrium is efficient if it maximizes the aggregate payoffs of the play-

[4] We may alternatively assume that the probability distribution used by Nature is any function of the set of players that did not form a coalition yet. Then the vector θ is substituted by a function $\theta(T)$ that assigns for every set of remaining players T a vector such that $\theta_i > 0$ for all $i \in T$ and $\sum_{i \in T} \theta_i = 1$. Making this assumption would not affect the main results in this paper.

[5] Formally, let R_t be the set of players who have not yet formed a coalition at time t. If $|R_t| \in \{0; 1\}$ and $|R_{t-1}| > |R_t|$, we say that a coalition structure has formed at time t.

ers. Efficiency in the game with discounting implies two requirements: immediate agreement and formation of the coalition structure with maximal aggregate payoffs.

We will denote the noncooperative game described above by $G(N;\varphi;\theta;\delta)$. We will also introduce notation for the reduced games. A reduced game is a subgame starting after a coalition has been formed. Once a certain set of players $N\setminus T$ have formed coalitions, bargaining continues with the players set T. Players in T play the reduced game $G(T;\varphi^{\pi_{N\setminus T}};\theta;\delta)$ where $\varphi^{\pi_{N\setminus T}}: E(T)\to\mathbb{R}$ is obtained from φ by fixing $\pi_{N\setminus T}$.

2.2.2 The game with breakdown probability

This game is identical to the game with discounting, except in the source of friction in the bargaining process. Players do not discount future payoffs but after a proposal is rejected Nature selects a new proposer with probability p (we will again think of p as being close to 1) and a breakdown of the negotiations occurs with probability $1-p$. If a breakdown of the negotiations occurs, all players that did not form a coalition yet remain singletons. Thus, if we denote the set of remaining players by T and the set of coalitions already formed by $\pi_{N\setminus T}$, a breakdown of the negotiations induces the partition $\pi_{N\setminus T}\cup\langle T\rangle$.

Payoffs in the game with breakdown probability are as in the previous game but undiscounted. Thus, when a coalition structure π is formed, coalitions containing more than one player share the coalitional payoff in the following way: player i whose proposal to form S was accepted pays $x_j^{S\setminus\{i\}}$ to responder j and gets the residual payoff $\varphi(S;\pi)-\sum_{j\in S\setminus\{i\}}x_j^{S\setminus\{i\}}$, and each singleton $\{k\}$ receives $\varphi(k;\pi)$. Unlike in the game with discounting, delay does not imply a loss of efficiency.

The possibility of breakdown implies that a coalition structure forms with probability 1 in this game no matter what strategies are played (either players agree to it, or breakdown will eventually occur).

We will denote the game with breakdown probability by $G(N;\varphi;\theta;p)$ and the reduced game arising after partition $\pi_{N\setminus T}$ has formed by $G(N;\varphi^{N\setminus T};\theta;p)$.

2.2.3 Expected payoffs and continuation values

Let σ be a combination of stationary strategies. Suppose that no coalitions have formed yet. Thus, we are at the beginning of the game or at a subgame that is equivalent to the beginning of the game. We will denote the expected payoff of player i given σ by $w_i(\sigma)$. This expectation is computed before Nature draws the proposer. We will denote by $w_i^j(\sigma)$ the expected payoff for player i given that player j has been selected to be the proposer.

Suppose now a proposal has been made to player i. The expected payoff of player i if he rejects a proposal is called the continuation value of player i and will be denoted by $z_i(\sigma)$: This is also i's expected payoff if somebody else other than i rejects a proposal.

In the game with discounting, it holds that $z_i(\sigma) = \delta w_i(\sigma)$. In the game with probability of breakdown, $z_i(\sigma)$ is a convex combination of $w_i(\sigma)$ (with weight p) and $\varphi(i; \langle N \rangle)$ (with weight $1 - p$).

Because the strategies are stationary, these values are the same at any subgame provided that no coalitions have formed yet. The definitions are analogous for a reduced game. We will denote player i's expected payoff in the reduced game arising after $\pi_{N \setminus T}$ has formed by $w_i^{\pi_{N \setminus T}}(\sigma)$, and his continuation value by $z_i^{\pi_{N \setminus T}}(\sigma)$: We will also drop σ from the notation when no confusion can arise.

2.2.4 The ordering of the responders and the identity of the proposer

The order in which the responders accept or reject a proposal need not be specified, since it has no practical relevance. In any subgame perfect equilibrium, a player accepts any proposal that gives all the responders at least their continuation values. If a proposal gives one of the responders less than his continuation value, it will be rejected (possibly by another responder). Since the consequences of a rejection are the same regardless of which player rejected the proposal, the order in which players respond to a proposal does not affect the results.

The identity of the proposer will also be of little relevance in the following sense: whether a payoff vector $x^{S \setminus \{i\}}$ will be accepted or rejected does not depend on the identity of the proposer i: What matters is how this payoff vector compares with the continuation values of the players in $S \setminus \{i\}$, and stationarity implies that the continuation values are independent of who made the last rejected proposal.

3 No-delay results

Given a strategy combination, we will say that a coalition structure forms without delay in equilibrium if strategies are such that all proposals that can be made with positive probability will be accepted. In all other cases (thus also in the extreme case when players never make acceptable proposals), we will speak of possible delay.

3.1 The game with discounting

Proposition 1 states that a coalition structure will form without delay if the underlying partition function is fully cohesive and positive. This proposition is an extension of theorem 1 in Okada (1996) to partition function games. The proof rests on the following lemma:

Lemma 1 Consider a positive and fully cohesive game $(N;\varphi)$. In any stationary perfect equilibrium σ^* of the game $G(N;\varphi;\theta;\delta)$ it holds that

(i) $\varphi(N) \geq \sum_{i\in N} w_i(\sigma^*)$.

(ii) $\varphi^{\pi_{N\setminus T}}(T) \geq \sum_{i\in T} w_i^{\pi_{N\setminus T}}(\sigma^*)$.

Proof. Since the game is positive and fully cohesive, the maximum aggregate payoff for the players is achieved if the grand coalition is formed immediately. Delay of the agreement or formation of subcoalitions can only reduce aggregate payoffs[6]. The same reasoning applies to any reduced game. ■

Proposition 1 Consider a positive and fully cohesive game $(N;\varphi)$. In any stationary perfect equilibrium σ^* of the game $G(N;\varphi;\theta;\delta)$, a coalition structure is formed without delay.

Proof. Consider any stationary subgame perfect equilibrium σ^*. We will denote by $\mu(\pi^t|(\sigma^*;S))$ the probability that coalition structure π is formed at time t given that players follow σ^* and that S forms. For every $i=1,\dots,n$, let $m_i(\sigma^*)$ be the maximum value of the following maximization problem

$$\max_{S;y} \sum_{\substack{\pi\in\Pi(N)\\ S\in\pi}} \sum_{t=1}^{\infty} \mu(\pi^t|(\sigma^*;S))\delta^{t-1}\left[\varphi(S;\pi) - \sum_{j\in S\setminus\{i\}} y_j\right]$$

$$\text{s.t. (i)} \quad i\in S\subseteq N$$

$$\text{(ii)} \quad \sum_{\substack{\pi\in\Pi(N)\\ S\in\pi}} \sum_{t=1}^{\infty} \mu(\pi^t|(\sigma^*;S))\delta^{t-1} y_j \geq \delta w_j(\sigma^*)$$

We will show that $w_i^i(\sigma^*) = m_i(\sigma^*)$, that is, the expected payoff for player i given that he is selected to be the proposer and follows his equilibrium strategy equals

[6] Notice that this result holds for any strategy combination, not necessarily an equilibrium.

the expected payoff for player i given that he makes the proposal $(S; y)$ that solves the maximization problem above.

Subgame perfection implies $w_i^i(\sigma^*) \geq m_i(\sigma^*)$. In a subgame perfect equilibrium all responders accept any proposal that gives each responder j at least $\delta w_j(\sigma^*)$ in expected terms, thus player i can get at least $m_i(\sigma^*)$.

Player i cannot get more than $m_i(\sigma^*)$. If he makes a proposal $(S; y^{S\setminus\{i\}})$ with $\sum_{\pi \in \Pi(N); S \in \pi} \sum_{t=1}^{\infty} \mu(\pi^t|(\sigma^*; S))\delta^{t-1}[\varphi(S; \pi) - \sum_{j \in S\setminus\{i\}} y_j] > m_i(\sigma^*)$, the proposal will be rejected (otherwise at least one responder j is getting less than $\delta w_j(\sigma^*)$ and could do better by rejecting the proposal) and i will get $\delta w_i(\sigma^*)$. Lemma 1 states that $\varphi(N) \geq \sum_{i \in N} w_i(\sigma^*)$. This inequality implies that player i could have proposed the grand coalition, paid every other player j $w_j(\sigma^*)$ and kept at least $w_i(\sigma^*)$ for himself. Thus, $m_i(\sigma^*) \geq w_i(\sigma^*) \geq \delta w_i(\sigma^*)$.

Since player i can always get $m_i(\sigma^*)$ (by making the optimal acceptable proposal) and he cannot get more than $m_i(\sigma^*)$ (unacceptable proposals yield at most $m_i(\sigma^*)$), it follows that $w_i^i(\sigma^*) = m_i(\sigma^*)$.

To prove that player i strictly prefers to make acceptable proposals, we must prove $\delta w_i(\sigma^*) < m_i(\sigma^*)$. Since $m_i(\sigma^*) \geq w_i(\sigma^*)$ and $\delta < 1$, $\delta w_i(\sigma^*) = m_i(\sigma^*)$ would imply $w_i(\sigma^*) = m_i(\sigma^*) = 0$. Since $\varphi(N) \geq \sum_{j \in N} w_j(\sigma^*)$, $m_i(\sigma^*) \geq (1 - \delta)\varphi(N) > 0$ (any player can get a strictly positive payoff as a proposer by exploiting the cohesiveness of the partition function and the impatience of the other players).

Notice that the same reasoning applies to any reduced game, so that the whole coalition structure forms without delay. ■

We have assumed that the proposer pays the responders after a coalition structure has formed. If instead he pays the responders immediately after S is formed, his objective function becomes $\sum_{\pi \in \Pi(N)} \sum_{t=1}^{\infty} \mu(\pi^t|(\sigma^*; S))\delta^{t-1}\varphi(S; \pi) - \sum_{j \in S\setminus\{i\}} y_j$, and the constraint becomes $y_j \geq \delta w_j(\sigma^*)$. Notice that the reasoning of the proof is also valid for this case.

Corollary 1 Consider a positive and fully cohesive game $(N; \varphi)$. In any stationary perfect equilibrium σ^* of the game $G(N; \varphi; \theta; \delta)$, every player i in N proposes a solution of the following maximization problem

$$\max_{S;y} \left[\sum_{\pi \in \Pi(N); S \in \pi} \mu(\pi|(\sigma^*; S)) \varphi(S; \pi) - \sum_{j \in S\setminus\{i\}} y_j \right] \tag{2}$$
$$\text{s.t.} \quad i \in S \subseteq N$$
$$y_j \geq \delta w_j(\sigma^*)$$

The proposer makes a proposal that maximizes his own payoff subject to the proposal being accepted. Since players are risk neutral, what matters for the proposer is his expected payoff. Player i's expected payoff depends on the equilibrium strategies of the other players. The expression for player i's expected payoff incorporates the fact that there is no delay in equilibrium.

Corollary 2 Consider a positive and fully cohesive game $(N;\varphi)$. In any stationary perfect equilibrium of the game $G(N;\varphi;\theta;\delta)$, every player i in N has a strictly positive expected payoff.

As we have seen in the proof of lemma 1, a player can get a strictly positive expected payoff as a proposer by exploiting the impatience of the other players together with the cohesiveness of the partition function. As a responder, he can guarantee himself a zero payoff by rejecting all proposals and possibly becoming a singleton.

Corollary 3 Consider a positive and fully cohesive game $(N;\varphi)$. Every player i gets a higher payoff as a proposer than as a responder.

This follows immediately from the fact that $m_i(\sigma^*) > \delta w_i(\sigma^*)$.

From the proof of proposition 1 we can rank the three following payoffs: the payoff a player gets as a proposer, his expected payoff before the proposer is selected, and his payoff as a responder, namely

$$m_i(\sigma^*) = w_i^i(\sigma^*) > w_i(\sigma^*) > w_i^j(\sigma^*) = \delta w_i(\sigma^*) .$$

We have assumed that the partition function is positive. This assumption plays an essential role in the proof of proposition 1. First, the grand coalition must have a strictly positive payoff or players would (weakly) prefer to bargain forever. Second, to make sure that the whole coalition structure is formed without delay, all subcoalitions must have a strictly positive payoff so that the argument can apply to every possible reduced game. Singletons are an exception since they are not always formed voluntarily (that is, if only one player is left the singleton coalition is formed automatically). The arguments in the proof would still go through if singletons could have negative payoffs, but we have chosen to keep the partition function positive.

We have limited the analysis to positive partition functions mostly for reasons of interpretation. Since the extensive form game is such that players do not receive payoffs until a coalition structure is formed, the use of this game to model partition function bargaining seems only appropriate for situations in which coalitions get positive payoffs. Imagine a situation of conflict: once a coalition is formed, it may be that the players outside the coalition can only get negative payoffs. Then these players would prefer not to reach an agreement, and no payoffs would be realized since a coalition structure has not been formed. This does not seem to be a feasible alternative in actual conflicts[7]. The result also shows that "normalization" of payoffs is not innocuous in this game (only normalization by multiplying by a constant is innocuous).

We include a counterexample that shows that delay may arise if the partition function is not positive. A counterexample that shows that delay may arise if the partition function is not fully cohesive can be found in section 3.3.

Example 5 Delay with a partition function that is not positive

$N = \{1, 2, 3\}$

$\pi(1; 2; 3) = (-1; -1; -1)$

$\pi(ij; k) = (-2; -2)$

$\pi(123) = -3$

This partition function is fully cohesive but not positive. There are many stationary perfect equilibria, all of them ending in perpetual disagreement. For example, all players may make unacceptable proposals (offering to each responder a payoff of less than zero). It may also be that the first player to be selected forms a singleton coalition, and then the other two players never reach an agreement.

3.2 The game with breakdown probability

In contrast to the game with discount factor, delay is not completely excluded in the game with breakdown probability. However, delay will not be a generic phenomenon. Even though players do not discount payoffs, the possibility of a breakdown will induce players to reach an agreement unless players have nothing to lose by the breakdown, which generically is not the case in fully cohesive games. Furthermore, possible delay implies efficiency in the subgame where it arises.

Notice that the eventual formation of a coalition structure is guaranteed by the structure of the game.

Remark 1 In any stationary perfect equilibrium of the game $G(N; \varphi; \theta; p)$ a coalition structure is eventually formed.

[7] One can assign utilities different from zero to the outcome of perpetual disagreement. However, this seems paradoxical: when are these payoffs realized?

Because a breakdown of the negotiations occurs with positive probability every time a proposal is rejected, a coalition structure will eventually form even if players never make acceptable proposals.

Notice that, unlike in the game with discounting, players cannot "escape" from negative payoffs by making unacceptable proposals. This is the reason why in this section we assume only that the game is fully cohesive, and not that the game is positive.

Lemma 2 Consider a fully cohesive game $(N; \varphi)$. In any stationary perfect equilibrium σ^* of the game $G(N; \varphi; \theta; p)$ it holds that

(i) $\varphi(N) \geq \sum_{i \in N} z_i(\sigma^*)$.

(ii) $\varphi^{\pi_{N\setminus T}}(T) \geq \sum_{i \in T} z_i^{\pi_{N\setminus T}}(\sigma^*)$.

Proof. Since the game is fully cohesive and a coalition structure always forms, the maximum total payoff for the players is achieved if the grand coalition is formed. The formation of subcoalitions (or the breakdown of the negotiations) can only reduce total payoffs[8]. The same reasoning applies to any reduced game $G(T; \varphi^{\pi_{N\setminus T}}; \theta; p)$. ■

Proposition 2 Consider a fully cohesive game $(N; \varphi)$. Suppose there is a stationary perfect equilibrium σ^*, a reduced game $G(T; \varphi^{\pi_{N\setminus T}}; \theta; p)$ on the equilibrium path and a player i in T such that player i makes unacceptable proposals with positive probability in $G(T; \varphi^{\pi_{N\setminus T}}; \theta; p)$. Recall that $z_i^{\pi_{N\setminus T}}(\sigma^*)$ is the continuation value of player i given that $\pi_{N\setminus T}$ has formed. Then

a) Player i could get the same payoff by making an acceptable proposal to the grand coalition (of the remaining players) T.

b) $\sum_{i \in T} z_i^{\pi_{N\setminus T}}(\sigma^*) = \varphi^{\pi_{N\setminus T}}(T)$.

Proof. For every $i = 1, ..., n$, recall that $w_i^j(\sigma^*)$ denotes player i's equilibrium expected payoff conditional on player j becoming the proposer at time 1. Analogously to the game with discounting, we denote by $m_i(\sigma^*)$ the maximum payoff the proposer can get given that he makes an acceptable proposal. In the game with breakdown probability, $m_i(\sigma^*)$ is the value of the following maximization problem

$$\max_{S,y} \sum_{\pi \in \Pi(N)} \mu(\pi | (\sigma^*; S))\, \varphi(S; \pi) - \sum_{j \in S\setminus\{i\}} y_j$$

$$\text{s.t. (i) } i \in S \subseteq N$$

$$\text{(ii) } y_j \geq z_j(\sigma^*)$$

[8] Notice that this result holds for any strategy combination, not necessarily an equilibrium.

where $\mu(\pi|(\sigma^*;S))$ is the probability that coalition structure π is eventually formed given that players follow σ^* and that S forms, and $z_j(\sigma^*)$ is the continuation value for player j, that is, the expected payoff for player j when he rejects a proposal. Notice that since payoffs are not discounted, the time at which a coalition structure is formed does not matter for payoffs. We show that $w_i^i(\sigma^*) = m_i(\sigma^*)$.

Subgame perfection implies $w_i^i(\sigma^*) \geq m_i(\sigma^*)$. In a subgame perfect equilibrium all responders accept any proposal that gives each responder j at least $z_j(\sigma^*)$ in expected terms, thus player i can get at least $m_i(\sigma^*)$.

On the other hand, player i cannot get more than $m_i(\sigma^*)$. If player i proposes $(S; y^{S\setminus\{i\}})$ with $\sum_{\pi\in\Pi(N)} \mu(\pi|(\sigma^*;S))\,\varphi(S;\pi) - \sum_{j\in S\setminus\{i\}} y_j > m_i(\sigma^*)$, the proposal will be rejected (otherwise at least one responder j is getting less than $z_j(\sigma^*)$ and could do better by rejecting the proposal) and i will get $z_i(\sigma^*)$. Since the game is cohesive, lemma 2 implies that $\varphi(N) \geq \sum_{j\in N} z_j(\sigma^*)$. Therefore, player i could have proposed the grand coalition, paid every other player j $z_j(\sigma^*)$ and kept at least $z_i(\sigma^*)$ for himself. Thus, $m_i(\sigma^*) \geq z_i(\sigma^*)$, and $m_i(\sigma^*) = z_i(\sigma^*)$ only if $\sum_{j\in N} z_j(\sigma^*) = \varphi(N)$.

The previous reasoning applies to delay in the formation of a coalition given that the set of remaining players is N. Since the game is fully cohesive, the same reasoning applies to the occurrence of delay in any reduced game. ■

Since player i weakly prefers to propose the grand coalition (of the remaining players) and offer each of the other players his continuation value, a stationary perfect equilibrium in which agreement is delayed is "weak".

Lemma 2 implies that if players would have lexicographic preferences (preferring to agree earlier than later holding everything else constant) then all stationary perfect equilibria would exhibit immediate agreement.

Since players do not discount the future, delay is not a source of inefficiency. Furthermore, possible delay in a subgame implies efficiency in the subgame as the following corollary shows.

Corollary 4 Consider a fully cohesive game $(N;\varphi)$. Suppose there is a reduced game $G(T;\varphi^{\pi_{N\setminus T}};\theta;\delta)$ exhibiting possible delay. Then

i) The final outcome is always efficient for this reduced game.

ii) $\sum_{i\in S} \varphi^{\pi_{N\setminus T}}(i;\langle T\rangle) = \varphi^{\pi_{N\setminus S}}(T)$.

Proof. We make the reasoning for the case of possible delay in the formation of the first coalition. The same reasoning applies to any reduced game.

The sum of the continuation values, $\sum_{i \in N} z_i$, is a convex combination of the sum of the expected payoffs given that Nature continues the game (with weight p) and the sum of expected payoffs if breakdown occurs (with weight $1 - p$). Each of these sums is at most $\varphi(N)$. Since possible delay implies $\sum_{i \in N} z_i = \varphi(N)$, in an equilibrium with delay both sums must be equal to $\varphi(N)$. Since players can never get more than $\varphi(N)$, $\sum_{i \in N} z_i = \varphi(N)$ implies that after a proposal is rejected equilibrium strategies are such that players always (and not only on average) get $\varphi(N)$. This must also be the case at the beginning of the game because of stationarity, thus efficiency is always achieved. In particular, the sum of expected payoffs if breakdown occurs, $\sum_{i \in N} \varphi(i; \langle N \rangle_i)$, equals $\varphi(N)$. ■

Corollary 4 implies that delay will not be a generic phenomenon, since it requires $\sum_{i \in S} \varphi^{\pi_{N \setminus S}}(i; \langle S \rangle_i) = \varphi^{\pi_{N \setminus S}}(S)$. This condition means that the total payoff of the players that remain in the game can never be higher than the total payoff they would obtain as singletons, so that the formation of coalitions can only bring inefficiency (though not in equilibrium).

Corollary 5 If the grand coalition cannot form in equilibrium, neither can delay occur.

This follows directly from the fact that in equilibrium a player weakly prefers to form the grand coalition rather than to make an unacceptable proposal.

3.3 An example of possible delay with a partition function that is not fully cohesive

This subsection provides an example of a partition function that is superadditive but not fully cohesive, and shows that delay is possible in equilibrium. Since the game with discounting and the game with breakdown probability are equivalent provided that $\varphi(i; \langle N \rangle_i) = 0$ for all $i \in N$, example 6 is valid for the game $G(N; \varphi; \theta; p)$ as well as for $G(N; \varphi; \theta; \delta)$. The proof uses the notation of the game with discounting (rephrasing the proof in the terms of the game with breakdown probability is straightforward).

Example 6 $N = \{1; 2; 3; 4; 5; 6; 7; 8; 9\}$, $\theta_i = \frac{1}{9}$ for all i.

We are going to consider a symmetric game. Since the number of players is large, we will depart from our usual notation and denote each coalition by its size.

$\varphi(4; 2; 3) = (16; 12; 9)$

$\varphi(6; 3) = (28; 2)$

$\pi(2;5;2) = (1;20;1)$
$\pi(5;4) = (21;3)$
$\pi(2;7) = (2;25)$
$\pi(9) = 30$

For the other coalition structures, it holds that, regardless of how the rest of the players are distributed, singletons get 0 and coalitions of size $s > 1$ get a payoff of s. Thus, for example, $\pi(6;2;1) = (6;2;0)$. Notice that this partition function is superadditive[9].

This example is based in the following idea: a coalition structure of the type $(4;2;3)$ is going to arise in equilibrium. In order for this to be the case, coalitions have to form in a given order: a coalition of size 2 is the most attractive in terms of per capita payoffs, but it cannot form first because then a coalition of size 5 would follow and the coalition of size 2 would have a low payoff. A coalition of size 4 would give raise to the coalition structure $(4;2;3)$, but it cannot form immediately, because then it would be more profitable in expected terms to wait until someone else forms the coalition of 4 and then be in the coalition of 2 or in the coalition of size 3 (the lottery of being either in a coalition of size 2 or in a coalition of size 3 is more attractive than being in the coalition of size 4 for sure). There is an asymmetric equilibrium in which some players form a coalition of 4 and others wait, thus delay is possible.

Consider the partition $(4;2;3)$ as a candidate equilibrium partition (in the calculations below, all the numerical values are computed for $\delta \to 1$). We will first consider symmetric equilibria (that is, equilibria in which all players propose a coalition of the same size and, in doing so, they propose to each of the other players with equal probability. This implies that, for any reduced game, all the players have the same expected payoff).

We first check that coalition structure $(4;2;3)$ cannot form in equilibrium starting from a coalition of size 2.

[9] The game in this example is not cohesive. It is easy to construct an example of possible delay in a cohesive game by using this example as a subgame. For example, consider a symmetric game with 20 players such that the first proposer will propose a coalition of size 11. Given that this coalition of 11 has formed, the subgame with 9 players is identical to the game in example 6. To make the game cohesive and to ensure that the coalition of 11 players will form, set $\pi(11;4;2;3) = (1100;16;12;9)$, and $\varphi(N) = 1137$ (and $\pi(15;2;3) = (1116;2;3)$, $\pi(11;6;3) = (11;28;2)$...).

Suppose the formation of a coalition of size 2 is followed by a coalition of size 4 and then by a coalition of size 3. Then the expected payoffs[10] (after the formation of a coalition of size 2) would be given by

$$w = \frac{1}{7}(16 - 3\delta w) + \frac{6}{7}\left[\frac{3}{6}\delta w + \frac{3}{6}3\right]$$

This equation reflects the fact that, when a player is selected to be the proposer (with probability $\frac{1}{7}$) he proposes a coalition of size 4. If the player is not selected to be the proposer (with probability $\frac{6}{7}$) he will be a responder with probability $\frac{3}{6}$ (since the proposer will form a coalition of size 4 and all the remaining players will have the same probability to be responders) and he will be left out with probability $\frac{3}{6}$. In this case, a coalition of the three remaining players would form without delay, so that each of the three players will have an expected payoff of 3.

The solution to the equation is $w = \frac{25}{7}$. The proposer gets then about $\frac{37}{7}$. However, he could do better by proposing a coalition of size 5 and, counting on the fact that the remaining two players will form a coalition, get $20 - 4 \times \frac{25}{7} = \frac{40}{7}$.

If a coalition of size 3 were to follow the coalition of size 2, this would not be an equilibrium since the players would rather wait than form the coalition of size 3. Expected payoffs (taking into account that a coalition of 4 will form after the coalition of 3) would be

$$w = \frac{1}{7}(9 - 2\delta w) + \frac{6}{7}\left[\frac{2}{6}\delta w + \frac{4}{6}4\right]$$

This yields $w = \frac{25}{7}$. The proposer would get $9 - 2 \times \frac{25}{7} = \frac{13}{7} < \frac{25}{7}$. He would then prefer to wait, hoping to be in a coalition of size 4 later on[11].

Suppose a coalition of 4 forms first, and this is followed by a coalition of size 2 and then by a coalition of size 3. It is clear that a coalition of size 3 would form, given that the two other coalitions have formed. It is easy to check that a coalition of size 2 would follow a coalition of size 4. Thus, $(4; 2; 3)$ is an equilibrium coalition structure provided that a coalition of size 4 forms in the first place. But will this be the case? Suppose a coalition of size 4 forms without delay. Then the expected payoff at the beginning of the game, w, would be given by the following equation

$$w = \frac{1}{9}[16 - 3\delta w] + \frac{8}{9}\left[\frac{3}{8}\delta w + \frac{5}{8}\left(\frac{2}{5}6 + \frac{3}{5}3\right)\right]$$

[10] We will denote all the expected payoffs by w (without subindex since it will be the same for all players and without superindex to simplify notation, hoping that no confusion will arise). We will also omit from the notation the strategy combination with respect to which expected payoffs are computed.

[11] In general, if the partition function and the protocol are symmetric and the equilibrium is such that players are to be divided into two coalitions with different per capita payoffs, the coalition with the highest per capita payoff should form first (or players would prefer to wait, hoping to be in the second coalition). See also section 6.3 on the role of per capita payoffs.

This yields $w = \frac{37}{9} > 4$. This cannot be an equilibrium for high values of $\pm$ since the proposer would get a higher payo¤ by making an unacceptable proposal.

The reason why forming a coalition of size 4 cannot be an equilibrium is that waiting (hoping to get into the coalition of size 2 later on) is a more attractive alternative.

Analogously, coalition structure $(4; 2; 3)$ cannot form starting by a coalition of size 3 because, even though a coalition of 2 and then a coalition of 4 would follow a coalition of 3, the coalition of 3 would not form in the …rst place.

There is an equilibrium in which coalitions of sizes $4; 2$, and 3 form in this order, but this equilibrium exhibits delay. Suppose four of the players make acceptable proposals to each other (of forming a coalition of size 4) and the other …ve "wait" in the hope of getting into a coalition of size 2. The two groups of players may now have di¤erent expected payo¤s, so we will denote them by w_l and w_h respectively:

$$w_l = \frac{1}{9}[16 ¡ 3\pm w_l] + \frac{8}{9}\pm w_l$$

$$w_h = \frac{5}{9}\pm w_h + \frac{4}{9}\left(\frac{2}{5}6 + \frac{3}{5}3\right)$$

This yields $w_l = \frac{16}{9 ¡ 5\pm}$ and $w_h = \frac{84}{5(9 ¡ 5\pm)}$.

We only have to check that a proposer whose continuation value is w_l prefers to propose the coalition of 4 (and get approximately 4). Alternatively, he could propose a coalition of 3 and get approximately $9 ¡ 8 = 1$, a coalition of size 2 and get a negative payo¤ (since a coalition of 5 will follow a coalition of 2), a coalition of 1 and get 0, a coalition of 5 or 6 and get a negative payo¤ (since this would be followed by a coalition of 3 and 2 respectively), a coalition of 7 and get 1, a coalition of 8 and get a negative payo¤, or the grand coalition and get $¡2$.

In fact, coalitions of size 4, 2 and 3 have to form in this order. A coalition of size 3 cannot be the …rst coalition to form, even with delay, because then players would prefer to form a coalition of 4 …rst.

One may also wonder whether there are asymmetric equilibria without delay. This does not seem to be the case. The reason is the following: we have managed to construct an equilibrium where some players form a coalition of 4 and some others wait. Thus, some players will be in a coalition of size 4 with probability 1 in equilibrium, and this is what makes their expected payo¤s close to $\frac{16}{4} = 4$. If we want both a coalition of 4 players forming …rst and no delay, all players have to propose a coalition of 4, but this implies that some player will be in the coalition of 4 with probability less than 1 and thus have expected payo¤s larger than 4 (since the expected payo¤ from being left out of the size 4 coalition,

$\frac{2}{5}6 + \frac{3}{5}3 = \frac{21}{5}$, is larger than the expected payoff from being in it[12], $\frac{16}{4}$) and players would prefer to make unacceptable proposals rather than follow their prescribed strategies.

4 Formation of the grand coalition

Formation of the grand coalition implies efficiency for fully cohesive games, since the other possible source of inefficiency (delay of the agreement in the game with discounting) has been excluded by the results in the previous section. We will provide sufficient conditions for the existence of a stationary perfect equilibrium in which the grand coalition forms with probability 1.

4.1 The game with discounting

The following lemma will be useful

Lemma 3 Suppose there is a stationary perfect equilibrium of $G(N;\varphi;\theta;\delta)$ in which the grand coalition forms with probability 1. Then the expected payoff for player i equals

$$w_i = \theta_i \varphi(N). \tag{3}$$

Proof. If all players propose the grand coalition, the expected payoff for player i can be found from the following equation

$w_i = \theta_i \left[\varphi(N) - \delta \sum_{j \in N \setminus \{i\}} w_j\right] + (1 - \theta_i)\delta w_i.$

Re-arranging terms yields

$w_i = \theta_i \left[\varphi(N) - \delta \sum_{j \in N} w_j\right] + \delta w_i.$

Using $\sum_{j \in N} w_j = \varphi(N)$, we obtain $w_i = \theta_i \varphi(N)$. ■

Proposition 3 Let $(N;\varphi)$ be a positive and fully cohesive game. There exists a stationary perfect equilibrium of the game $G(N;\varphi;\theta;\delta)$ in which the grand coalition forms with probability 1 if

$$\sum_{j \in S} \theta_j \varphi(N) \geq \varphi(S;\pi) \text{ for all } (S;\pi) \in E(N). \tag{4}$$

[12] We use the fact that all players must have the same expected payoffs in the reduced game.

Proof. Suppose all players propose the grand coalition. Expected payoffs are then given by (3). This strategy combination is an equilibrium if no proposer can do better by proposing some other coalition S. If the proposer proposes to form the grand coalition, his payoff is

$\varphi(N) - \delta \sum_{j \in N \setminus \{i\}} \theta_j \varphi(N).$

If he proposes some other coalition S instead, his payoff depends on the coalition structure that eventually forms. A sufficient condition for a deviation not to be profitable is that it is not profitable for any coalition structure, thus

$\varphi(N) - \delta \sum_{j \in N \setminus \{i\}} \theta_j \varphi(N) \geq \varphi(S;\pi) - \delta \sum_{j \in S \setminus \{i\}} \theta_j \varphi(N)$ for all $(S;\pi) \in E(N)$

Re-arranging terms, we obtain

$\varphi(N) - \delta \sum_{j \in N \setminus S} \theta_j \varphi(N) \geq \varphi(S;\pi)$ for all $(S;\pi) \in E(N)$.

If we want this condition to hold for all δ, it is sufficient that it holds for $\delta = 1$. Substituting $\delta = 1$ and re-arranging, we obtain

$\sum_{j \in S} \theta_j \varphi(N) \geq \varphi(S;\pi)$ for all $(S;\pi) \in E(N)$. ■

Corollary 6 *If all players are selected to be the proposers with the same probability, the sufficient condition (4) becomes*

$$\frac{\varphi(N)}{|N|} \geq \frac{\varphi(S;\pi)}{|S|} \text{ for all } (S;\pi) \in E(N). \tag{5}$$

Condition (5) means that the grand coalition has the highest per capita payoff of all possible embedded coalitions.

4.2 The game with probability of breakdown

Lemma 4 *Suppose there is a stationary perfect equilibrium of $G(N;\varphi;\theta;p)$ in which the grand coalition forms with probability 1. Then the expected payoff for player i equals*

$$w_i = \varphi(i;\langle N \rangle) + \theta_i \left[\varphi(N) - \sum_{j \in N} \varphi(j;\langle N \rangle) \right] \tag{6}$$

and his continuation value equals

$$z_i = \varphi(i;\langle N \rangle) + p\theta_i \left[\varphi(N) - \sum_{j \in N} \varphi(j;\langle N \rangle) \right]. \tag{7}$$

Proof. If all players propose the grand coalition, the continuation value for player j can be found from the following equation

$z_i = p\theta_i(\varphi(N) - \sum_{j \in N \setminus \{i\}} z_j) + p(1 - \theta_i)z_i + (1 - p)\varphi(i; \langle N \rangle).$

Re-arranging terms yields

$z_i = p\theta_i(\varphi(N) - \sum_{j \in N} z_j) + pz_i + (1 - p)\varphi(i; \langle N \rangle).$

Substituting for $\sum_{j \in N} z_j = p\varphi(N) + (1 - p)\sum_{j \in N} \varphi(j; \langle N \rangle)$ we obtain

$z_i = \varphi(i; \langle N \rangle) + p\theta_i(\varphi(N) - \sum_{j \in N} \varphi(j; \langle N \rangle)).$

Expected payoffs are given by

$w_i = \theta_i(\varphi(N) - \sum_{j \in N \setminus \{i\}} z_j) + (1 - \theta_i)z_i.$

Re-arranging terms and substituting for z_i, we get

$w_i = \varphi(i; \langle N \rangle) + \theta_i(\varphi(N) - \sum_{j \in N} \varphi(j; \langle N \rangle)).$ ■

Proposition 4 Let $(N; \varphi)$ be a fully cohesive game. There exists a stationary perfect equilibrium of the game $G(N; \varphi; \theta; p)$ in which the grand coalition always forms if

$$\sum_{j \in S} \theta_j \left[\varphi(N) - \sum_{j \in N} \varphi(j; \langle N \rangle) \right] \geq \varphi(S; \pi) - \sum_{j \in S} \varphi(j; \langle N \rangle) \;\; \forall (S; \pi) \in E(N). \quad (8)$$

Proof. Suppose all players propose the grand coalition. Then the payoff a player gets as a responder is given by (7) and the payoff he gets as a proposer is given by

$\varphi(N) - p\sum_{j \in N \setminus \{i\}} \theta_j \left[\varphi(N) - \sum_{j \in N} \varphi(j; \langle N \rangle) \right] - \sum_{j \in N \setminus \{i\}} \varphi(j; \langle N \rangle))$.

If player i proposes another coalition S, his payoff will depend of the coalition structure that eventually forms. Again, it is sufficient that a deviation will not be profitable for any coalition structure.

$\varphi(N) - p\sum_{j \in N \setminus \{i\}} \theta_j \left[\varphi(N) - \sum_{j \in N} \varphi(j; \langle N \rangle) \right] - \sum_{j \in N \setminus \{i\}} \varphi(j; \langle N \rangle)) \geq$

$\varphi(S; \pi) - p\sum_{j \in S \setminus \{i\}} \theta_j \left[\varphi(N) - \sum_{j \in N} \varphi(j; \langle N \rangle) \right] - \sum_{j \in S \setminus \{i\}} \varphi(j; \langle N \rangle))$ for all $(S; \pi) \in E(N).$

Notice that it is sufficient that this inequality holds for $p = 1$. Substituting $p = 1$, substracting $\theta_i \left[\varphi(N) - \sum_{j \in N} \varphi(j; \langle N \rangle) \right] - \varphi(i; \langle N \rangle))$ from both sides and rearranging we get

$\sum_{j \in S} \theta_j \left[\varphi(N) - \sum_{j \in N} \varphi(j; \langle N \rangle) \right] \geq \varphi(S; \pi) - \sum_{j \in S} \varphi(j; \langle N \rangle) \;\; \forall (S; \pi) \in E(N).$ ■

Corollary 7 If each player is selected to be the proposer with the same probability, the sufficient condition (8) becomes

$$\frac{\varphi(N) - \sum_{i\in N}\varphi(i;\langle N\rangle)}{|N|} \geq \frac{\varphi(S;\pi) - \sum_{i\in S}\varphi(i;\langle N\rangle)}{|S|} \text{ for all } (S;\pi) \in E(N). \quad (9)$$

Condition (9) means that the grand coalition has the maximum per capita gain with respect to the situation in which all players are singletons.

4.3 Some examples

The examples in this subsection show that neither of the two extensive form games considered so far promotes efficiency better than the other. This is true not only for a fixed protocol but also if the protocol can be chosen so as to maximize the efficiency of the outcome.

Example 7 Consider the following partition function with $N = \{1;2;3\}$

$\varphi(1;2;3) = (1;2;3)$
$\varphi(12;3) = (4;0)$
$\varphi(1;23) = (0;6)$
$\varphi(13;2) = (5;0)$
$\varphi(123) = 8$

We may think of the players as three firms differing in efficiency. A coalition can then be interpreted as a merger, a (binding) agreement to collude, a research joint venture, etc. The (internal) profit gain from any merger of two firms is 1, whereas the profit gain form forming the grand coalition is 2. The game is fully cohesive and the only efficient coalition structure is the grand coalition.

Suppose that each of the players is selected to be the proposer with the same probability, thus $\theta_i = \frac{1}{3}$ for $i = 1;2;3$: The sufficient condition (9) holds (thus always forming the grand coalition is an equilibrium in the game with probability of breakdown), whereas the sufficient condition (5) does not hold. We now check that forming the grand coalition with probability 1 is an equilibrium in the game with probability of breakdown, but not in the game with discounting.

Suppose all players propose the grand coalition. In the game with probability of breakdown, each player's expected payoff equal his payoff in the event of breakdown plus an equal share of the gain from forming the grand coalition (equation (6)). Thus, each player gains $\frac{2}{3}$ with respect to the situation in which all players are singletons.

$w_1 = 1 + \frac{2}{3}; w_2 = 2 + \frac{2}{3}; w_3 = 3 + \frac{2}{3}$

In the limit when p tends to 1, each player receives w_i regardless of whether he was the proposer or the responder. Deviations to other strategies will not be profitable: any pair of players gains more ($\frac{4}{3}$) by forming the grand coalition than by forming a two-player coalition (1).

Things are different in the game with discounting. Equation (3) prescribes equal shares of the value of the grand coalition, thus expected payoffs (and actual payoffs in the limit when $\delta \to 1$) equal $\frac{8}{3}$. Unlike in the game with breakdown probability, these strategies do not constitute an equilibrium. Player 2 prefers to propose to player 3, offer him $\frac{8}{3}$ and keep for himself $6 - \frac{8}{3} > \frac{8}{3}$.

Since expected payoffs given that the grand coalition forms are sensitive to the probabilities of being a proposer, we may find a different probability vector that, by favoring players 2 and 3, will give them an incentive to stick to the grand coalition.

Consider, for example, $\mu = (\frac{1}{8}; \frac{3}{8}; \frac{4}{8})$. If all players propose the grand coalition, expected payoffs equal $w_1 = 1$, $w_2 = 3$ and $w_3 = 4$. Always forming the grand coalition is now an equilibrium.

Example 8 Consider the following partition function

$\phi(1; 2; 3) = (5; 0; 0)$

$\phi(12; 3) = (5; 0)$

$\phi(1; 23) = (0; 4)$

$\phi(13; 2) = (5; 0)$

$\phi(123) = 8$

Player 1 is a very productive player when all players are alone. However, the cooperation of player 1 with one of the other two players does not bring any additional value. Cooperation of the other two players against player 1 is very profitable (they earn four units more than they were earning separately) and cooperation of the three players is a bit less profitable (they earn three units).

Suppose all players are selected to be proposers with the same probability. None of the sufficient conditions is satisfied, so that it is a priori unclear whether formation of the grand coalition will be an equilibrium. We will check that this is the case in the game with discounting, but not in the game with possible breakdown.

Suppose the grand coalition always forms in the game with discounting. This implies that each player has an expected payoff (and actual payoff in the limit when δ tends to 1) of $\frac{8}{3}$. This is an equilibrium even though player 1 receives less than he got when all players were alone. The reason is that player 1 cannot secure 5 for himself: if he decides to stay alone players 2 and 3 will form a coalition and player 1 will get zero. On the other hand, no player will profit from proposing a two-player coalition since any two players get $\frac{16}{3} > 5$.

All players proposing the grand coalition cannot be an equilibrium in the game with breakdown probability for any protocol. The reason is that player 1 must receive at least 5. Player 2 and 3 together get then no more than 3, so that any of them has an incentive to propose coalition $\{2,3\}$ instead of the grand coalition.

It seems that the game with impatience promotes efficiency better if we are allowed to chose the protocol, since we can induce any division of $\varphi(N)$ by manipulating the vector θ, whereas in the game with breakdown probability we are constrained by the fact that each player must receive at least $\varphi(i; N \setminus i)$. This is however not the case since the payoff a coalition can expect when deviating from the grand coalition may be different for the two games, so that temptation to defect may be higher in the game with discounting. The following example illustrates this fact.

Example 9 $N = \{1,2,3,4\}$

$v(12,3,4) = (25,1,5)$

$v(12,34) = (20,8)$

$v(ijk,l) = (30,0)$

$\varphi(N) = 40$

$v(\pi) = (0,\ldots,0)$ for all other $\pi \in \Pi(N)$.

An efficient equilibrium is not possible in the game with discounting for any protocol. In order to achieve an efficient outcome, the protocol has to be symmetric, since each three-player coalition can get a payoff of 30, and this together with $\varphi(N) = 40$ implies that each player has to get exactly 10. The only candidate protocol for an efficient equilibrium is then the egalitarian protocol.

Consider the game with discounting. If coalition $\{1,2\}$ forms, players 3 and 4 will not form a two-player coalition for sure because this would imply a payoff of 4 for each player and then player 4 would prefer to form a singleton. Thus, players 1 and 2 together must receive more than 20 in an efficient equilibrium, but this is not feasible with an egalitarian protocol.

In the game with breakdown probability, the symmetric protocol achieves an efficient outcome. Coalition $\{1,2\}$ cannot receive more than 20, since the formation of this coalition will be followed by the formation of coalition $\{3,4\}$.

5 Efficient equilibria and the core

Chatterjee et al. (1993) show that the limit payoff vector of an efficient equilibrium must belong to the core if the underlying characteristic function game is strictly superadditive. This implies that no efficient equilibrium can arise (for high discount factors) if the underlying game is strictly superadditive and has an empty core. The following proposition establishes a similar result for strictly cohesive partition function games[13]: in an efficient equilibrium, the expected payoff vector belongs to the core of a characteristic function game that assigns to each coalition its expected payoff given the equilibrium strategies. We will denote this characteristic function simply by v^*. Notice however that v^* is a function of the protocol θ and of the extensive form game.

Proposition 5 *Let $(N;\varphi)$ be a strictly cohesive partition function game. Suppose there is a sequence $\delta^k \to 1$ and a corresponding sequence of efficient stationary perfect equilibria $\sigma^*(\delta^k)$ of the game $G(N;\varphi;\theta;\delta^k)$. Then the expected payoff vector $w = (w_i(\sigma^*(\delta^k)))_{i\in N}$ does not depend on δ^k and is in the core of the characteristic function game $(N;v^*)$, where*

$$v^*(S) = \lim_{\delta\to 1} \sum_{\substack{\pi\in\Pi(N)\\ S\in\pi}} \rho(\pi|(\sigma^*(\delta);S))\,\varphi(S;\pi).$$

Proof. Efficiency implies that the grand coalition forms immediately regardless of which player is selected to be the proposer. Then expected payoffs are $w_i = \theta_i\varphi(N)$ for all $i\in N$. Since agreement is immediate, expected payoffs do not depend on δ.

Suppose now that player i is selected to be the proposer in equilibrium. If he sticks to his prescribed strategy and proposes to form the grand coalition, he offers δw_j to each player $j\in N\setminus\{i\}$ and keeps $\varphi(N) - \sum_{j\in N\setminus\{i\}}\delta w_j$ for himself. In order for this to be an equilibrium strategy, player j should not have a profitable deviation, thus

$$\varphi(N) - \sum_{j\in N\setminus\{i\}}\delta w_j \ge \sum_{\substack{\pi\in\Pi(N)\\ S\in\pi}}\sum_{t=1}^{\infty}\rho(\pi^t|(\sigma^*;S))\,\delta^{t-1}\varphi(S;\pi) - \sum_{i\in S\setminus\{i\}}\delta w_j \quad \forall S\ni j. \tag{10}$$

[13] A partition function game $(N;\varphi)$ is strictly cohesive if $\varphi(N) > \sum_{S\in\pi}\varphi(S;\pi)$ for all $(S;\pi)\in E(N)$. Analogously to strict superadditivity in characteristic function games, strict cohesiveness implies that the only efficient coalition structure is the grand coalition.

Since any player may be selected to be the proposer, condition (10) must be satisfied for each $j \in N$.

In the limit when $\delta \to 1$ the advantage of the proposer disappears and each player i gets w_i regardless of whether he is a proposer or a responder. Condition 10 becomes

$\sum_{i \in S} w_i \geq v^*(S)$ for all $S \subset N$

or, substituting for w_i,

$\sum_{i \in S} \theta_i \varphi(N) \geq v^*(S)$

that is, the vector w must be in the core of the game $(N; v^*)$. ■

Proposition 5 illustrates the assumptions of the model about the reaction of the complement of S if S forms. If S forms, the complement of S does not necessarily react in such a way that the payoff of S is minimized. The coalition structure that forms given S is an equilibrium coalition structure (there may be several possible coalition structures if the equilibrium is in mixed strategies) and the payoff S receives is a subgame perfect equilibrium payoff. Thus, no incredible threats on the part of $N \setminus S$ are assumed.

A similar proposition holds for the game $G(N; \varphi; \theta; p)$. The difference is in the expected payoff vector w and in the equilibrium strategy vector σ^*.

Proposition 6 *Let $(N; \varphi)$ be a strictly cohesive partition function game. Suppose there is a sequence $p^k \to 1$ of continuation probabilities and a corresponding sequence of efficient stationary perfect equilibria $\sigma^*(p^k)$ of the game $G(N; \varphi; \theta; p^k)$ with expected payoff vector $w = (w_i(\sigma^*(p^k)))_{i \in N}$. Then w is independent of p^k and lies in the core of the characteristic function game $(N; v^*)$, where*

$$v^*(S) = \lim_{p \to 1} \sum_{\substack{\pi \in \Pi(N) \\ S \in \pi}} \mu(\pi|(\sigma^*(p); S))\, \varphi(S; \pi).$$

Proof. Efficiency implies that the grand coalition always forms regardless of which player is selected to be the proposer. The continuation values are then given by

$z_i = \theta_i p(\varphi(N) - \sum_{j \in N} \varphi(j; \langle N \rangle)) + \varphi(i; \langle N \rangle)$

and expected payoffs are given by

$w_i = \theta_i(\varphi(N) - \sum_{j \in N} \varphi(j; \langle N \rangle)) + \varphi(i; \langle N \rangle)$

Each player j gets then an expected payoff of z_i as a responder and $\varphi(N) - \sum_{j \neq i} z_j > z_i$ as a proposer. In the limit when p tends to 1, the advantage of the proposer disappears and each of the players gets w_i. For the strategy combination σ^* to be an equilibrium, we then need

$\sum_{i\in S} w_i \geq v^*(S)$ for all $S \subset N$:

or, substituting for w_i;

$\sum_{i\in S} \theta_i \left[\varphi(N) - \sum_{j\in N} \varphi(j;\langle N\rangle) \right] + \sum_{i\in S} \varphi(i;\langle N\rangle) \geq v^*(S)$: ■

Nonemptiness of the core of v^* is not sufficient for the game to have an efficient equilibrium. This is easy to see for the game with breakdown probability, since the payoff for a player i in an efficient equilibrium is constrained to be at least $\varphi(i;\langle N\rangle)$, a value that may not be relevant for the computation of v^*. The following example illustrates this point.

Example 10 We compute the function v^* for the partition function in example 8. $v^*(123) = 8; v^*(12) = v^*(13) = 5; v^*(23) = 4$ and $v^*(i) = 0$ for all i:

The function is the same for both extensive form games, but this is not the case in general[14]. Notice that the formation of the grand coalition or a two-player coalition completely determines the coalition structure. For singleton coalitions, one has to consider the reaction of the players external to the coalition. If player 3 forms a singleton, players 1 and 2 will form a two-player coalition, thus $v^*(3) = 0$. If players 1 or 2 form a singleton, their payoff is 0 regardless of whether the other two players form a coalition, thus $v^*(1) = v^*(2) = 0$.

The core of this game is nonempty but there is no efficient equilibrium in the game $G(N;\varphi;\theta;p)$ for any θ. If there was an efficient equilibrium for some θ, w_1 would be at least 5, but no payoff vector that gives player 1 at least 5 lies in the core.

Nonemptiness of the core is sufficient for the existence of an efficient equilibrium in the game with discounting provided that the game v^* does not change with the protocol. However, it will generally be the case that v^* changes with the protocol, and then nonemptiness of the core for a given protocol does not imply the existence of an efficient equilibrium. The following example illustrates this point.

Example 11 $N = \{1;2;3;4\}$

$\varphi(14;2;3) = (1;0;0)$

$\varphi(13;2;4) = (1;7;0)$

$\varphi(2;134) = (0;1)$

$\varphi(3;124) = (0;1)$

$\varphi(4;123) = (0;8)$

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[14] See example 9, in which $v^*(123)$ is larger for the game with discounting given an egalitarian protocol.

$\rightharpoonup(12;34) = \rightharpoonup(14;23) = (1;20)$
$\rightharpoonup(13;24) = (1;7)$
$\rightharpoonup(N) = 24$
$\rightharpoonup(¼) = (0;:::;0)$ for all other ¼:
Note that this game is fully cohesive.

We now calculate the function v¤ for the game G(N; ' ; µ; p) with $µ_i = \frac{1}{4}$ for all i. Since the coalition structure is completely determined by the formation of a three-player coalition or the grand coalition, v¤(N) = 24, v¤(123) = 8, v¤(124) = v¤(134) = v¤(134) = 1 regardless of the extensive form game. Moreover, players 1, 3 and 4 get a payo¤ of 0 in all coalition structures where they are singletons, thus v¤(1) = v¤(3) = v¤(4) = 0 regardless of the extensive form game. If player 2 forms a singleton, the other three players …nd themselves in a situation where only coalitions including player 1 are pro…table. It is easy to check that players 2 and 3 will propose to player 1 and that, given that the protocol is egalitarian, player 1 will propose to each of the other two players with equal probability, therefore coalitions f1; 3g and f1; 4g form with probability $\frac{1}{2}$ each, and v¤(2) = 3:5. If coalition f1; 3g forms, it gets 1 regardless of the coalition structure. Any other two-player coalition will trigger the coalition of the complement, thus v¤(12) = v¤(14) = 1, v¤(24) = 7, and v¤(23) = v¤(34) = 20.

The core of this game is nonempty for the egalitarian protocol: for example, (0; 4; 17; 3) is in the core.

A necessary condition for the game to have an e¢cient equilibrium for some protocol is that the game v¤ associated to the protocol has a nonempty core. Notice that v¤(23) and v¤(34) do not depend on the protocol. Thus, given any protocol, a payo¤ vector has to satisfy the following necessary conditions in order to be in the core of the corresponding v¤:

$$w_2 + w_3 \quad ¸ \quad 20 \text{ (which implies } 4 \; ¸ \; w_4) \qquad (11)$$
$$w_4 + w_3 \quad ¸ \quad 20 \text{ (which implies } 4 \; ¸ \; w_2): \qquad (12)$$

Thus, w_3 ¸ 16 in any e¢cient equilibrium. This in turn implies $µ_3 \; ¸ \; \frac{16}{24} = \frac{2}{3}$, but $µ_3 \; ¸ \; \frac{2}{3}$ implies v¤(2) ¸ $\frac{14}{3} > 4$, contradicting (12).

6 Random proposers versus rule of order

Ray and Vohra (1999) study a noncooperative game in which the …rst player to reject a proposal becomes the next proposer. This implies that the order in which players accept or reject proposals may affect the results, so that one has to specify a rule of order selecting not only proposers but also responders. Ray and Vohra assume that players discount payoffs, so that the comparison will be made between their game and the game $G(N,\varphi,\theta,\delta)$. We will assume $\theta_i = \frac{1}{n}$ unless otherwise specified.

The game with a rule of order does not guarantee immediate agreement for fully cohesive games. The conditions for the formation of the grand coalition are robust to changes in the rule of order, unlike in the game with random proposers, where we obtained different conditions for each protocol. If players are symmetric, the game with a rule of order ensures that payoff division inside a coalition is symmetric as well, unlike in the game with random proposers. If players are asymmetric, the game with a rule of order may give too much power to the responders, so that competition between players is not reflected in the outcome. As for the efficiency of the outcome, the two procedures cannot be ranked in general. The two procedures are equivalent for symmetric games with (symmetric) fixed payoff division.

6.1 No delay result

In the context of characteristic function games, Okada (1996) proves that superadditivity of the underlying characteristic function implies no delay in the game with random proposers, unlike in the game with a rule of order considered by Chatterjee et al. (1993). Analogously, one can check that full cohesiveness does not guarantee immediate agreement in the game considered by Ray and Vohra (1999). Indeed, any superadditive characteristic function is a fully cohesive partition function, so that the result follows immediately.

Consider the following example, quoted in Chatterjee et al. (1993) and Okada (1996) and originally due to Bennett and van Damme:

Example 12 $N = \{1,2,3,4\}, v(\{1,j\}) = 50$, $j = 2,3,4$; $v(\{i,j\}) = 100$, $i,j = 2,3,4$, $v(S) = 100$, $|S| = 3$ and $v(N) = 150$.

This example can be rephrased in terms of a partition function, where $\phi(\langle N\rangle) = (0,0,0,0)$; $\phi(1i,jk) = (50,100)$, $i,j,k = 2,3,4$; $\varphi(1i,j,k) = (50,0,0)$, i, j, $k = 2,3,4$; $\phi(1,i,jk) = (0,0,100)$, $i,j,k = 2,3,4$; $\phi(ijk,l) = (100,0)$, $i,j,k,l = 1,2,3,4$; $\phi(N) = 150$.

There are three strong players and a weak player in this example (in fact, the core consists of one single point: $(0,50,50,50)$). If the rule of order is such that player 1 starts the game, he will make an unacceptable proposal. The reason is as follows: if a player other than player 1 is selected to be the proposer, he will propose to other of the "strong" players. It is easy to see that all strong players have the same continuation value, $\frac{100\delta}{1+\delta}$ (that is, almost 50 for large δ). If player 1 makes his best acceptable proposal (the grand coalition) he gets $150 - \frac{300\delta}{1+\delta}$, close to 0 for large δ. If instead he makes an unacceptable proposal to a strong player, this player will reject and form a coalition with another strong player. This leaves player 1 and the remaining strong player in a symmetric situation, so that player 1 can get a payoff of about 25 rather than a payoff close to 0.

In the game with random proposers, player 1 will make an acceptable proposal to the grand coalition. Expected payoffs (for the case $\mu_i = \frac{1}{4}$ for all i) are then given by the following system of equations, where w_s represent the expected payoff for a strong player[15]

$w_1 = \frac{1}{4}(150 - 3\delta w_s) + \frac{3}{4}25$

$w_s = \frac{1}{4}(100 - \delta w_s) + \frac{1}{2}\delta w_s + \frac{1}{4}25$

The solution to this system is (in the limit when $\delta \to 1$) $w_1 = 25$ and $w_s = \frac{125}{3}$. The reason why there is no delay is that the game with random proposer gives less power to the responders. When a strong player rejects an offer, he is not sure of being the next proposer; the continuation values reflect this risk, so that it is profitable for the weak player to make an acceptable proposal rather than to wait.

6.2 Formation of the grand coalition

The sufficient conditions we found in section 4 were sensitive to the protocol. However, in the game with a rule of order, one obtains condition (5) regardless of the rule of order. Thus, the game with a rule of order yields more "robust" results, while the game with random proposers is more "flexible". In example 7, one could obtain an efficient outcome by changing the protocol, whereas the game with a rule of order does not have an efficient equilibrium for any rule of order.

[15] It's easy to see that all strong players must have the same expected payoff in equilibrium.

6.3 Formation of the coalition with highest per capita payoff in symmetric games

Ray and Vohra show that, for symmetric games and provided that the equilibrium exhibits no delay, players form the coalition that maximizes the average per capita payoff given the reaction of outsiders.

This is not always the case in the game $G(N; \varphi; \theta; \delta)$ (even with a symmetric protocol), as the following example shows

Example 13 $N = \{1; 2; 3; 4\}$, $v(4) = 18$, $v(3; 1) = (14; 0)$, $v(\pi) = 0$ for all other π.

This is a game in characteristic function, so we do not need to consider the reaction of outsiders.

The coalition of size 3 has the highest per capita payoff. Suppose however that players propose a coalition of size 3 in the game with random proposers. Then the expected payoff (given a symmetric protocol) is given by

$w = \frac{1}{4}(14 - 2\delta w) + \frac{3}{4}\frac{2}{3}\delta w$

This yields $w = 3.5$. The payoff of the proposer is then approximately 7. However, he would get an even higher payoff by proposing the grand coalition $(18 - 3 \times 3.5 = 7.5)$.

The reason why larger coalitions that have lower per capita payoffs instead of smaller coalitions that have higher per capita payoffs form is that a player who rejects a proposal will not be the next proposer for sure. Thus, he will be left out of the coalition that eventually forms with positive probability, and this negatively affects his continuation value. Since the responders are "underpaid" because of the risk they have of being left out if they reject the proposal, it may pay to form larger coalitions. Example 13 thus points to a trade-off between efficiency and distribution.

In the game with a rule of order, players in the same coalition always split the payoff equally (see Ray and Vohra, 1999). Thus, responders are never "underpaid".

Per capita payoffs also play some role in the game with random proposers. For example, suppose in a symmetric game that the equilibrium is such that exactly two coalitions will form without delay. Then, it is still true that the coalition with the higher per capita payoff must form first, or, if both coalitions have the same per capita payoff, the largest coalition must form first (otherwise players would prefer to wait instead of forming the coalition with the smallest per capita payoff).

6.4 Competition between responders

As we have seen in the previous subsection, responders are never "exploited" in symmetric games with a rule of order. Indeed, when players are asymmetric, responders may have "too much power", as in the following market game (cf example 2 in Chatterjee et al.)

Example 14 $N = \{1, 2, ..., n\}$. Player 1 is a seller who owns a unit of some good and players $2, ..., n$ are potential buyers whose reservation price is u. The characteristic function is such that $v(S) = 1$, $|S| \geq 2$, $1 \in S$, and $v(S) = 0$ otherwise.

We now calculate the continuation value of a buyer in the game with a rule of order. After rejecting a proposal, he will make a proposal to the seller and offer him his continuation value. Thus, $z_i = \delta(1 - z_1)$. This continuation value is the same for all buyers regardless of the strategy of the seller. Let us denote it by z_b. No matter how the seller randomizes between the potential buyers, his continuation value will be $z_1 = \delta(1 - z_b)$. Therefore, $z_1 = z_b = \frac{1}{1+\delta}$, or the seller cannot benefit from the competition between the buyers.

Instead, in a game with random proposers, the equilibrium would be much more competitive: two buyers are enough to drive the price up to 1 (in the limit when δ tends to 1). Expected payoffs would be then given by the following system of equations

$$w_1 = \frac{1}{n}(1 - \delta w_b) + \frac{n-1}{n}\delta w_1$$
$$w_b = \frac{1}{n}(1 - \delta w_1) + \frac{1}{n}\frac{1}{n-1}\delta w_b$$

These equations take into account that the seller must propose to each of the potential buyers with the same probability. The reason is that, on the one hand, the seller would like to propose to the buyer with the lowest continuation value, whereas, on the other hand, the expected payoff of a buyer is larger the larger is the probability that he receives a proposal from the seller.

Expected payoffs are then $w_1 = \frac{n-1-\delta}{n(n-1)-\delta(n^2-2n+2)}$ and $w_b = \frac{(n-1)(1-\delta)}{n(n-1)-\delta(n^2-2n+2)}$.

It can be checked that the payoff to the seller increases with n. Even with only two players, the price converges to 1 in the limit when $\delta \to 1$.

6.5 Efficiency of the outcome

Proposition 1 and example 13 seem to point in the direction of higher efficiency for the game with random proposers. This is indeed the case for very specific games (like three-person quota games with the grand coalition and symmetric games without externalities where only one coalition can form), but not in general (see example 15 below).

6.5.1 Three-person quota games with the grand coalition

Consider the following cooperative game. $N = \{1;2;3\}; v(i) = 0 \ \forall i \in N$, $v(1;2) = a + b$, $v(1;3) = a + c$, $v(2;3) = b + c$, $v(1;2;3) = a + b + c$, $a > b > c$.

The equilibrium of the game with a rule of order is as follows. Players 1 and 2 propose to each other and split nearly equally, and player 3 proposes to player 1. Continuation values are $z_1 = z_2 = \delta\frac{a+b}{1+\delta}$, $z_3 = \delta c + \delta\frac{a-\delta b}{1+\delta}$. Given these continuation values, nobody wishes to propose the grand coalition, since this would imply adding a responder whose continuation value is larger than his quota[16].

As for the game with random proposers, notice the following. As in the game with a rule of order, each proposer wishes to include all responders whose continuation value is less than their quota. Notice also that the sum of continuation values is strictly smaller than the sum of the quotas (that is, $\delta \sum_{i \in N} w_i < v(N) = a + b + c$). Thus, at least one player has a continuation value of less than his quota. This player must be a responder with probability 1. On the other hand, it cannot be the case that all players have a continuation value smaller than their quotas, since this would imply the grand coalition forming with probability 1 and all players splitting equally, contradicting $\delta w_3 < c$. Thus, at least one player and at most two must have a continuation value smaller than their quotas. Suppose only one player (player 1) satisfies this property. It can be checked that this is not possible when the other two players have continuation values that are larger than their quotas. It is possible to find an equilibrium in which the other two players have a continuation value that exactly equals their quota. Consider the following strategies: player 1 always proposes the grand coalition. Players 2 and 3 randomize between proposing the grand coalition and proposing to player 1. This strategy combination constitutes an equilibrium. In the limit when δ tends to 1, the probability of the grand coalition being formed tends to 1 as well. Thus, efficiency is higher in the game with random proposers.

[16] Experimental evidence supporting this outcome (though using a very different bargaining procedure) has been recently found by Bolton et al (1999).

6.5.2 Symmetric games without externalities where only one coalition can form

Consider a symmetric characteristic function game in which at most one coalition with positive value can form. Let k be the cardinality of the coalition with the highest per capita payoff (or, if there are several, of the largest coalition). This is the coalition that will form in the game with a rule of order. We now show that a coalition with a smaller total payoff cannot form with positive probability in the game with random proposers.

Suppose we have an equilibrium of the game with random proposers. Symmetry of the game and of the protocol imply that all players must have the same continuation value. Recall that, in the game with random proposers, the continuation value of a player is δ times his expected payoff. If we represent the value of a coalition of cardinality m by $v(m)$, and the probability of a coalition of size m to be formed given the (equilibrium) strategies of the players by λ_m, expected payoffs are

$$\frac{v(k)}{k} \geq \frac{\sum_{m=1}^{n} \lambda_m v(m)}{n}.$$

Since $\frac{v(k)}{k} \geq w > 0$; it follows that $\frac{v(k)}{k} > \delta w$.

A coalition of cardinality l with $v(l) < v(k)$ cannot form in equilibrium in the game with random proposers. The reasons are obvious for $l > k$. For $l < k$, the proposer would always want to enlarge the coalition to k players, since doing so will increase the value of the coalition by at least $(k - l)\frac{v(k)}{k}$, while he will only have to pay $\delta(k - l)w$.

Things are very different in games where more than one coalition with positive value can form. Consider the following game

Example 15 $N = \{1; 2; 3; 4; 5\}; v(1) = 0; v(2) = 5; v(3) = 14; v(4) = 18; v(5) = 19.$

In the game with a rule of order, a coalition of size 3 will form, followed by a coalition of size 2. Total payoffs are then 19. However, in the game with random proposers, a four-player coalition is formed with probability 1, so that total payoffs are only 18.

Thus, one can conclude that, except for very specific situations (like three-person quota games with the grand coalition or games in which only one coalition is formed), neither of the procedures (rule of order or random proposers) yields generally more efficient results.

6.6 Random proposers versus rule of order with fixed payoff division

A distinctive feature of the game with random proposers is that it puts responders in a weak position, and this induces them to accept lower payoffs than in the game with a rule of order. This suggests that in a game with fixed payoff division (such as the games considered by Bloch (1996)) it should not make a difference how proposers are selected.

Consider the following example:

Example 16 $N = \{1,2,3\}$

$v(i,j,k) = (0,0,0)$

$v(ij,k) = (12,3)$

$v(N) = 15$

Suppose payoff division is restricted to be egalitarian. Both in the game with a rule of order and in the game with random proposers, a two-player coalition forms in any subgame perfect equilibrium. The differences between the two games is limited to subgames that are off the equilibrium path. Should the grand coalition be proposed in the game with a rule of order, it would be rejected, since the rejector can earn $6 > 5$ by proposing a two-player coalition. In the game with random proposers, however, the grand coalition would be accepted, since, by rejecting the grand coalition, the responder is running the risk of being left out, and thus his expected payoff from rejecting the proposal is only (given a symmetric protocol) $\frac{2}{3}6 = 4 < 5$. Nevertheless, the grand coalition will not be proposed in equilibrium, so that selecting proposers at random makes no practical difference.

Drawing the proposer at random can make a difference in symmetric games if the (fixed) payoff division is not egalitarian. Suppose that, in the previous example, payoff division for the grand coalition must be $(6.8, 4.1, 4.1)$. Clearly, player 1 would like to form the grand coalition given that he gets more than in any other coalition structure. In the game with a rule of order, he cannot have his way. In the game with random proposers he can, at least if the protocol is asymmetric. Consider the protocol $\mu = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and the following strategies: player 1 always proposes the grand coalition, players 2 and 3 propose to each other with probability $\frac{4}{15}$ and to player 1 with probability $\frac{11}{15}$. This strategy combination is an equilibrium (since it yields $w_1 < 6$ and $w_2, w_3 < 4.1$).

7 Conclusion

We have studied two games of coalition formation with externalities and random proposers, di¤ering in the source of friction. There are two potential sources of ine¢ciency in those games: delay of the agreement (only in the game with discounting) and formation of subcoalitions. The su¢cient conditions we have found for e¢ciency are rather demanding. Analogously to the case of characteristic function games, the possibility of making binding agreements and the fact that there are gains from merging do not guarantee the formation of the grand coalition, even with perfect information.

We have also compared the model with random proposers to the model with a rule of order. This two models di¤er in the power of the responder. Giving less power to responders makes immediate agreement easier, though the …nal outcome is not necessarily more e¢cient. When players are symmetric, the game with a rule of order yields a symmetric payo¤ division whereas random proposer typically yield an asymmetric distribution. In some cases, this asymmetric distribution is needed to achieve e¢ciency, so that there is a trade-o¤ between e¢ciency and distribution. The same feature of the game with a rule of order that is attractive for symmetric games (equal payo¤ division due to the power of the responder) may be undesirable when players are asymmetric, since competition between players is not re‡ected in the outcome.

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